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SIMULTANEOUS APPROXIMATION PROPERTIES DE LA VALLÉE-POUSSIN MEANS IN MUSIELAK- ORLICZ SPACES

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Abstract: We investigate the simultaneous approximation properties De la Vallée-Poussin means in Musielak-Orlicz spaces in terms of the modulus of smoothness. In terms of the modulus of smoothness the direct theorem of simıltaneous approximation is proved. Also in Musielak-Orlicz spaces the modulus of smoothness are estimated from below and above in terms $n-$th partial sums and De la Vallée-Poussin means.

Keywords: Musielak-Orlicz spaces, Modulus of smoothness, $n-$th partial sums, De la Vallée-Poussin means, Best approximation

Subject Classification: 41A10, 41A50, 42A10.

1. Introduction

Let T denote the interval $[-\pi,\pi]$. A function $\varphi:[0,\infty]\to[0,\infty]$ is called $\Phi-$ *function* (briefly $\varphi\in\Phi$) if φ is convex and left –continuous and

$$\varphi(0):=\lim_{t\to0^+}\varphi(t)=0, \qquad \lim_{x\to\infty}\varphi(x)=\infty .$$

A $\Phi-$function φ is said to be an $N-$ function if it is continuous and positive and satisfies

$$\lim_{t\to0^+}\frac{\varphi(t)}{t}=0 , \ \lim_{t\to\infty}\frac{\varphi(t)}{t}=\infty .$$

Let $\Phi(\mathsf{T})$ be the collection of functions $\varphi:\mathsf{T}\times[0,\infty]\to[0,\infty]$ such that

(i) $\varphi(x,\cdot)\in\Phi$ for every $x\in\mathsf{T}$;

(ii) $\varphi(x,u)$ is in $L^0(\mathsf{T})$ $x\in\mathsf{T}$; the set of measurable functions, for every $u\geq0$.

A function $\varphi(\cdot,u)\in\Phi(\mathrm{T})$ is said to satisfy the Δ_2 condition $(\varphi\in\Delta_2)$ with respect to parameter u if $\varphi(x,2u)\le K\varphi(x,u)$ holds for all $x\in\mathrm{T}$, $u\ge 0$, with some constant $K\ge 2$.

Subclass $\Phi(N)\subset\Phi(\mathrm{T})$ consist of functions $\varphi\in\Phi(\mathrm{T})$ such that, for every $x\in\mathrm{T}$, $\varphi(x,\cdot)$ is an $N-$ function and $\varphi\in\Delta_2$.

We use $c,c_1,c_2,...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest.

2. Some Auxiliary Results And Main Results

Two functions φ and φ_1 are said to be *equivalent* (we shall write $\varphi\sim\varphi_1$) if there $c>0$ such that

$$\varphi_1(x,\frac{u}{c})\le\varphi(x,u)\le\varphi_1(x,cu)$$

for all x and u.

For $\varphi\in\Phi(N)$ we set

$$\rho_\varphi(f):=\int_{\mathrm{T}}\varphi(x,|f(x)|)dx.$$

Musielak – Orlicz space L^φ (or generalized Orlicz space) is the class of Lebesgue measurable functions $f:\mathrm{T}\to\mathbb{R}$ satisfying the condition

$$\lim_{\lambda\to 0}\rho_\varphi(\lambda f)=0.$$

The equivalent condition for $f\in L^0(T)$ to belong to L^φ is that $\rho_\varphi(\lambda f)<\infty$ for some $\lambda>0$. L^φ becomes a normed space with the Orlicz norm

$$\|f\|_{[\varphi]}:=\sup\left\{\int_{\mathrm{T}}|f(x)g(x)|dx:\rho_\psi(g)\le 1\right\}$$

and with the Luxemburg norm

$$\|f\|_\varphi:=\inf\left\{\lambda>0:\rho_\varphi\left(\frac{f}{\lambda}\right)\le 1\right\}$$

where

$$\psi(t,v):=\sup_{u\ge 0}(uv-\varphi(t,u)),v\ge 0,\ t\in\mathrm{T}$$

is the complementary function (with respect to variable v) of φ in the sense of Young. These two norms are equivalent :

$$\| f\|_{\varphi} \le \| f\|_{[\varphi]} \le 2\| f\|_{\varphi}$$

Young's inequality

$$us \le \varphi(x,u)+\psi(x,s)\ , \tag{2.1}$$

holds for complementary functions $\varphi,\ \psi \in \Phi(N)$ where $u\,, s \ge 0$ and $x \in \mathrm{T}$.

From Young's inequality (1.1) we have

$$\| f\|_{[\varphi]} \le \rho_{\varphi}(f)+1\ ,$$

$\| f\|_{\varphi} \le \rho_{\varphi}(f)$ if $\| f\|_{\varphi} > 1$ and $\| f\|_{\varphi} \ge \rho_{\varphi}(f)$ if $\| f\|_{\varphi} \le 1$.

Hölder's inequality

$$\int_{\mathrm{T}} \left| f(x)g(x)\right| dx \le \| f\|_{\varphi} \| f\|_{[\varphi]}$$

holds for complementary functions $\varphi,\ \psi \in \Phi(N)$. The Jensen integral inequality can be formulated as follows.

If φ is an $N-$ function and $r(x)$ is a nonnegative measurable function, then

$$\varphi\left(\frac{1}{\int_{\mathrm{T}} r(x)dx} \int_{\mathrm{T}} f(x)r(x)dx \right) \le \frac{1}{\int_{\mathrm{T}} r(x)dx} \int_{\mathrm{T}} \varphi(f(x)r(x))dx$$

Everywhere in this work we will assume that there exists a constant $A > 0$ such that for all $x,\ y \in \mathrm{T}$ with $|x-y| \le \frac{1}{2}$ we have

$$\frac{\varphi(x,u)}{\varphi(y,u)} \le u^{\frac{A}{\log(\frac{1}{|x-y|})}},\ u \ge 1 \tag{2.2}$$

there exist some constants $c_1, c_2 > 0$ such that

$$\inf_{x\in \mathrm{T}} \varphi(x,1) \ge c_1 \tag{2.3}$$

and

$$\int_{\mathrm{T}} \varphi(x,1)dx < \infty,\ \psi(x,1) \le c_2 \quad \text{a.e on } \mathrm{T}. \tag{2.4}$$

As can be seen from the definitions above, Musielak – Orlicz spaces are similar to Orlicz spaces but are defined by a more general function with two variables $\varphi(x,t)$. In these spaces, the norm is given by virtue of the integral

$$\int_{\mathrm{T}} \varphi(x, |f(x)|)dx\ ,$$

It is know that in an Orlicz space, φ would be independent of x, $\varphi(|f(x)|)$. The special cases $\varphi(t)=t^p$ and $\varphi(x,t)=t^{p(x)}$ give the Lebesgue spaces L^p and the variable exponent Lebesgue spaces $L^{p(x)}$, respectively. In addition to being a natural generalization that covers results from both variable exponent and Orlicz spaces, the study of Musielak – Orlicz spaces can be motivated by applications to differential equations $[12,36]$, fluid dynamics $[13,33]$, and image processing $[5,7,16]$. Detalied information on Musielak – Orlicz spaces can be found in the book by Musielak $[34]$

Example 2.1. *Let* $p:\mathsf{T}\to[1,\infty]$ *be in* $L^0(\mathsf{T})$ *such that for all* $x,y\in\mathsf{T}$ *with* $|x-y|\le\frac{1}{2}$ *we have the Dini – Lipschitz property.*

$$|p(x)-p(y)|\le\frac{c_3}{\log\left(\frac{1}{|x-y|}\right)},$$

with a constant $c_3>0$. *Then the following function belong to* $\Phi(T)$ *and satisfy conditions* (2.2), (2.3) *and* (2.4)

$$(i)\,\varphi(x,u)=u^{p(x)},\ \sup_{x\in\mathsf{T}}p(x)<\infty,$$

$$(ii)\,\varphi(x,u)=u^{p(x)}\log(1+u),$$

$$(iii)\,\varphi(x,u)=u(\log(1+u))^{p(x)}.$$

A function $\varphi\in\Phi(N)$ is in the class $\Phi(N,DL)$ if conditions (2.2), (2.3) *and* (2.4) are fulfilled.

For $f\in L^{\varphi}$ we define the Steklov operator A_h by

$$v_h(f)(x)=\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}f(x-t)dt,\ 0<h<\pi,\ x\in\mathsf{T}.$$

The characteristic function $K_{[a,b]}(u)$ of a finite interval $[a,b]$ is the function on defined thourgh

$$K_{[a,b]}(u)=\begin{cases}1,\ u\in[a,b],\\ 0,\ u\notin[a,b].\end{cases}$$

The operator v_h can be written as a convolution integral $[6,p.33],[4]$:

$$v_h(f)(x)=\frac{1}{2\pi}\int_{\mathsf{T}}f(t)R_h(t-x)dt,$$

where

$$R_h(u):=\frac{2\pi}{h}K_{\left[-\frac{h}{2},\frac{h}{2}\right]}(u).$$

Note that the kernel R_h satisfies the following conditions [6, p.33], [4]:

$$\int_{\mathrm{T}} R_h(u)du \le c_4\ ,\ \left|R_h(u)\right| \le c_5\ ,\ h \le u \le \pi \text{ and } \max_u \left|R_h(u)\right| \le c_6\frac{1}{h}.$$

Let $f \in L^{\varphi}$ and $\varphi \in \Phi(N, DL)$. By reference [4, Lemma 2], the shift operator v_h is a bounded linear operator on L^{φ}:

$$\left\|v_h(f)\right\|_{\varphi} \le c\left\|f\right\|_{\varphi}.$$

The function

$$\Omega_{\varphi}^{l}(\delta, f):=\sup_{0<h_i\le\delta}\left\|\prod_{i=1}^{l}(I-v_{h_i})f\right\|_{\varphi}\ ,\ \delta>0\ \ l=1,2,3,...$$

is called $l-th$ *order modulus of smoothness* of $f \in L^p\left(\mathrm{T}\right)$, where I is the identity operator.

It can easily be shown that $\Omega_{\varphi}^{l}(\cdot, f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega_{\varphi}^{l}(\delta, f)=0\ ,\ \Omega_{\varphi}^{l}(\delta, f+g)\le\Omega_{\varphi}^{l}(\delta, f)+\Omega_{\varphi}^{l}(\delta, g)$$

for $f, g \in L^{\varphi}$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k(x, f)=\sum_{k=-\infty}^{\infty}c_k e^{ikx} \tag{2.5}$$

be the Fourier series of the function $f \in L_1(\mathrm{T})$, where $A_k(x, f):=(a_k(f)\cos kx+b_k(f)\sin kx)$, $k=1,2,3,...,$ $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathrm{T})$.

The $n-th$ *partial sums*, and *De la Vallée – Poussin means* [53] of series (2.5) are defined, respectively as

$$S_n(f):=S_n(x, f)=\frac{a_0}{2}+\sum_{k=1}^{n}A_k(x, f)=\sum_{k=-n}^{n}c_k e^{ikx},\ n=1,2,3,...$$

$$V_n(f):=V_n(x, f)=\frac{1}{n}\sum_{v=n}^{2n-1}S_v(x, f).$$

Note that for the De la Vallée – Poussin means the integral representation

$$V_n(f):=V_n(x,f)=\int_{-\pi}^{\pi} f(x-t)K_n(t)dt,$$

holds with kernel

$$K_n(t):=\frac{1}{\pi}\frac{\sin(\frac{3nt}{2})\sin(\frac{nt}{2})}{2n\sin^2(\frac{t}{2})}.$$

The best approximation to $f\in L^{\varphi}$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{\varphi}:=\inf\left\{\left\|f-T_n\right\|_{\varphi}:T_n\in\Pi_n\right\}.$$

Note that the existence of $T_n^*\in\Pi_n$ such that

$$E_n(f)_{\varphi}=\left\|f-T_n^*\right\|_{\varphi}$$

follows, for example, from Theorem 1.1 in [9, p.59].

Let $W_{\varphi}^r\ (r=1,2,3,...)$, $\varphi\in\Phi(N,DL)$ be the class of functions such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)}\in L^{\varphi}$ becomes a Banach space under the consideration of the norm

$$\left\|f\right\|_{W_{\varphi}^r}:=\left\|f\right\|_{\varphi}+\left\|f^{(r)}\right\|_{\varphi}.$$

We use $c,c_1,c_2,...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest.

In the proof of the main results we need the following results.

Theorem 2.1. [4]. *For every* $f\in W_{\varphi}^r\ (r\in\mathbb{N})$, $\varphi\in\Phi(N,DL)$ *and* $n\in\mathbb{N}$ *the inequality*

$$E_n(f)_{\varphi}\le\frac{c_7}{(n+1)^r}E_n(f^{(r)})_{\varphi}$$

holds with a constant $c_7>0$ *depending only on* φ *and* r.

Theorem 2.2. [4]. *Let* $f\in L^{\varphi}$, $\varphi\in\Phi(N,DL)$ *and* $n\in\mathbb{N}$. *Then the estimate*

$$E_n(f)_{\varphi}\le c_8\Omega_{\varphi}^l(f,\frac{1}{n+1})$$

holds with a constant $c_8>0$ *depending only on* φ *and* r.

Using Theorem 2.1 and 2.2 we have the following Corollary:

Corollary 2.1. *For every* $f\in W_{\varphi}^r(r\in\mathbb{N})$, $\varphi\in\Phi(N,DL)$ *and* $n\in\mathbb{N}$ *the inequality*

$$E_n(f)_\varphi \le \frac{c_9}{(n+1)^r}\Omega_\varphi^l(f^{(r)},\frac{1}{n+1})$$

holds with a constant $c_9 > 0$ *depending only on* φ *and* r.

Lemma 2.1. [4]. *Let* $f \in L^\varphi, \varphi \in \Phi(N, DL)$ *and* $n \in \mathbb{N}$. *Then for each trigonometric polynomial* T_n *of degree* n *the inequality*

$$\left\|(T_n)^{(r)}\right\|_\varphi \le c_{10} n^r \left\|T_n\right\|_\varphi \qquad , \quad r = 1, 2, 3, ...$$

holds with a constant $c_{10} > 0$ *depending only on* φ *and* r.

Using the method of proof of [44, Theorem 2.1] and Theorem 2.2 we can prove the following Theorem:

Theorem 2.3. *Let* $f \in W_\varphi^r \ (r \in \mathbb{N}), \varphi \in \Phi(N, DL)$ *and* $n \in \mathbb{N}$. *Then the inequality*

$$\| f - V_n(f)\|_\varphi \le \frac{c_{11}}{(n+1)^r}\Omega_{M.w}^l(f^{(r)},\frac{1}{n+1})$$

holds with a constant $c_{11} > 0$ *depending only on* φ *and* r.

Theorem 2.4. *Let* T_n^* *be the best approximation polynomial to* f. *Then for every* $f \in W_\varphi^r \ (r = 0, 1, 2, ...)$, $\varphi \in \Phi(N, DL)$ *and* $n \in \mathbb{N}$ *the inequality*

$$\left\| f^{(r)} - (T_n^*)^{(r)}\right\|_\varphi \le c_{12} E_n(f^{(r)})_\varphi$$

holds with a constant $c_{12} > 0$ *depending only on* φ *and* r.

Proof of Theorem 2.4. We set

$$B_n(f) := B_n(x, f) = \frac{1}{n+1}\sum_{\nu=n}^{2n} S_\nu(x, f),\ n = 0, 1, 2, ...$$

Since

$$B_n(., f^{(\alpha)}) = B_n^{(\alpha)}(., f),$$

we have

$$\begin{aligned} &\left\| f^{(\alpha)}(\cdot) - T_n^*(\cdot, f)\right\|_\varphi \\ &\le \left\| f^{(\alpha)}(\cdot) - B_n(\cdot, f^{(\alpha)})\right\|_\varphi \\ &+\left\|T_n^{(\alpha)}(\cdot, B_n(f)) - T_n^{(\alpha)}(\cdot, f)\right\|_\varphi \\ &+\left\|B_n^{(\alpha)}(\cdot, f) - T_n^{(\alpha)}(\cdot, B_n(f))\right\|_\varphi \\ &= I_1 + I_2 + I_3 \ . \end{aligned} \qquad (2.6)$$

Let $T_n(x,f)$ be the best approximiting polynomial of degree at most n to f nin L_φ. From the boundedness of B_n in L^φ we have

$$I_1 \le \left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot,f) \right\|_\varphi + \left\| T_n^{(\alpha)}(\cdot,f) - B_n(\cdot, f^{(\alpha)}) \right\|_\varphi$$

$$\le c_{14} E_n(f^{(\alpha)})_\varphi + \left\| B_n^{(\alpha)}(\cdot, T_n(f^{(\alpha)})) - f^{(\alpha)} \right\|_\varphi \le c_{15} E_n(f^{(\alpha)})_\varphi \quad 2.7)$$

and by Lemma 2.1

$$I_2 \le c_{16} n^\alpha \left\| T_n(\cdot, B_n(f)) - T_n(\cdot, f) \right\|_\varphi \tag{2.8}$$

and

$$I_3 \le c_{17}(2n)^\alpha \left\| B_n(\cdot, f) - T(\cdot, B_n(f)) \right\|_\varphi$$

$$\le c_{18}(2n)^\alpha E_n(B_n(f))_\varphi . \tag{2.9}$$

The following inequalities hold:

$$\left\| T_n(\cdot, B_n(f)) - T_n(\cdot, f) \right\|_\varphi$$

$$\le \left\| T_n(\cdot, B_n(f)) - B_n(\cdot, f) \right\|_\varphi$$

$$+ \left\| B_n(\cdot, f) - f(\cdot) \right\|_\varphi + \left\| f(\cdot) - T_n(\cdot, f) \right\|_\varphi$$

$$\le c_{19} E_n(B_n(f))_\varphi . + c_{20} E_n(f)_\varphi + c_{21} E_n(f)_\varphi , \tag{2.10}$$

$$E_n(B_n(f))_\varphi . \le c_{22} E_n(f)_\varphi . \tag{2.11}$$

Use of (2.8), (2.9) and (2.11) gives us

$$I_2 \le c_{23} n^\alpha E_n(f)_\varphi , \tag{2.12}$$

$$I_3 \le c_{24} (2n)^\alpha E_n(f)_\varphi . \tag{2.13}$$

Taking into account the relations (2.6), (2.7), (2.12) and (2.13) we get

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_\varphi$$

$$\le c_{25} E_n(f^{(\alpha)})_\varphi + c_{26} n^\alpha E_n(f)_\varphi + c_{27}(2n)^\alpha E_n(B_n(f))_\varphi .$$

$$\le E_n(f^{(\alpha)})_\varphi + c_{28} n^\alpha E_n(f)_\varphi . \tag{2.14}$$

According to Theorem 2.1 the relation

$$E_n(f)_\varphi \le \frac{c_{29}}{(n+1)^\alpha} E_n(f^{(\alpha)})_\varphi \tag{2.15}$$

holds. Using (2.14) and (2.15) we have

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_\varphi \le c_{30} E_n(f^{(\alpha)})_\varphi .$$

The proof of Theorem 2.4 is completed.

Note that polynomial approximation problems in Musielak – Orlicz spaces have a long history. Orlicz spaces, which satisfy the translation invariance property, are a particular case of Musielak – Orlicz spaces. In these spaces, polynomial approximation problems were investigated by several mathematicians in [1, 8, 10, 18-20, 23-28, 31, 32, 39, 40, 52, 55, 56]. In general, Musielak – Orlicz spaces may not attain the translation invariance property, as can be seen in the case of variable exponent Lebesgue spaces $L^{p(x).}$. Several inequalitiers of trigonometric polynomial approximation in $L^{p(x).}$ were obtained in [2, 3, 15, 17, 43, 45]. Note that, under the translation invariance hypothesis on Musielak – Orlicz space, Musielak obtained some trigonometric approximation inequalities in [35].

In the present paper we investigate the simultaneous approximation properties of De la Vallee – Poussin means in Musielak – Orlicz spaces in terms $l-th$ order modulus of smoothness. Also , we estimate the modulus of smoothness from below and above in terms $n-th$ partial sums and De la Vallée – Poussin means in Musielak-Orlicz spaces. Similar problems in different spaces have been investigated by several researchers (see, for example ,[11, 14, 21, 22, 29, 30, 37, 38, 41, 42, 46-51, 54, 57]).

Our main results are as follows.

Theorem 2.5. *Let* $f \in W_{\varphi}^{r}\ (r \in \mathbb{N}),\ \varphi \in \Phi(N, DL),,\ m = 0,1,2,...,\ r$ *and* $n \in \mathbb{N}$. *Then the estimate*

$$\left\| f^{(m)} - V_n^{(m)}(f) \right\|_{\varphi} \le \frac{c_{31}}{n^{r-m}} \Omega_{\varphi}^{l}\left(\frac{1}{n}, f^{(r)}\right)$$

holds with a constant c_{31} *>0 independent of n.*

Theorem 2.6. *Let* $f \in L^{\varphi}$, $\varphi \in \Phi(N, DL)$ *and* $n \in \mathbb{N}$. *Then the following inequalities hold:*

1.

$$c_{32}\Omega_{\varphi}^{l}\left(\frac{1}{n}, f\right) \le \left(n^{-2l}\left\|V_n^{(2l)}(f)\right\|_{\varphi} + \left\| f - V_n(f)\right\|_{\varphi}\right)$$

$$\le c_{33}\Omega_{\varphi}^{l}\left(\frac{1}{n}, f\right), \tag{2.16}$$

where the constants c_{32} *and* c_{33} *independent of n..*

2.

$$c_{34}\Omega_{\varphi}^{l}\left(\frac{1}{n}, f\right) \le \left(n^{-2l}\left\|S_n^{(2l)}(f)\right\|_{\varphi} + \left\| f - S_n(f)\right\|_{\varphi}\right)$$

$$\le c_{35}\Omega_{\varphi}^{l}\left(\frac{1}{n}, f\right), \tag{2.17}$$

where the constants c_{34} and c_{35} independent of n.

3. Proofs of the main results

Proof of Theorem 3.1. Let $f \in W_{\varphi}^{r}$ and $T_n^{*} \in \Pi_n$ $(n = 0,1,2,...)$ be the polynomial of best approximation to f. The following inequality holds:

$$\left\| f^{(m)} - V_n^{(m)}(f) \right\|_{\varphi}$$

$$\leq \left\| f^{(m)} - (T_n^{*})^{(m)} \right\|_{\varphi} + \left\| (T_n^{*})^{(m)} - V_n^{(m)}(f) \right\|_{\varphi} \tag{3.1}$$

By virtue of Theorem 2.2 and 2.3 we get

$$\left\| f^{(m)} - (T_n^{*})^{(m)} \right\|_{\varphi} \leq c_{36} E_n(f^{(m)})_{\varphi}$$

$$\leq \frac{c_{37}}{n^{r-m}} E_n(f^{(r)})_{\varphi} \leq \frac{c_{38}}{n^{r-m}} \Omega_{\varphi}^{l}\left(\frac{1}{n}, f^{(r)}\right). \tag{3.2}$$

On the other hand using Lemma 2.1, Theorem 2.3 and Corollary 2.1 we obtain that

$$\left\| (T_n^{*})^{(m)} - V_n^{(m)}(f) \right\|_{\varphi}$$

$$\leq c_{39} n^{m} \left\{ \left\| V_n(f) - f \right\|_{\varphi} + \left\| f - T_n^{*} \right\|_{\varphi} \right\}$$

$$\leq c_{40} n^{m} \left\{ \frac{c_{16}}{n^{r}} \Omega_{\varphi}^{l}\left(\frac{1}{n}, f^{(r)}\right) + E_n(f)_{\varphi.} \right\}$$

$$\leq \frac{c_{41}}{n^{r-m}} \Omega_{\varphi}^{l}\left(\frac{1}{n}, f^{(r)}\right). \tag{3.3}$$

use of (3.1) , (3.2) and (3.3) , gives us

$$\left\| f^{(m)} - T_n^{(m)} \right\|_{\varphi} \leq \frac{c_{42}}{n^{r-m}} \Omega_{\varphi}^{l}\left(\frac{1}{n}, f^{(r)}\right)$$

The proof of Theorem 2.5 is compled.

Proof of Theorem 3.1. According to [4] the inequality

$$\Omega_{\varphi}^{l}\left(\frac{1}{n}, V_n(f)\right) \leq c_{43} n^{-2l} \left\| V_n^{(2l)}(f) \right\|_{\varphi} \tag{3.4}$$

holds. Taking into account the properties of modulus of smoothness $\Omega_{\varphi}^{l}(\frac{1}{n}, f)$ and (3.4) we conclude that

$$\Omega_\varphi^l\left(\frac{1}{n}, f\right)$$
$$\leq\left(\Omega_\varphi^l(\frac{1}{n}, f-V_n(f))+\Omega_\varphi^l(\frac{1}{n},V_n(f))\right)$$
$$\leq c_{44}\left(\left\| f-V_n(f)\right\|_\varphi+n^{-2l}\left\|V_n^{(2l)}(f)\right\|_\varphi\right). \qquad (3.5)$$

We estimate the modulus of smoothness $\Omega_{M,w}^l(\cdot, f)$ from below. Considering Theorem 2.1 and [4] the following inequalities hold:

$$E_n(f)_\varphi \leq c_{45}\Omega_\varphi^l\left(\frac{1}{n+1}, f\right), \qquad (3.6)$$

$$n^{-2l}\left\|V_n^{(2l)}(f)\right\|_\varphi \leq c_{46}\Omega_\varphi^l\left(\frac{1}{n+1}, f\right). \qquad (3.7)$$

Let $V_n(f,x)$ be the de la Vallée – Poussin sums of the series (2.5) and let $T_n^* \in \Pi_n$ be the polynomial of best approximation to f in L_φ, that is $\left\| f-T_n^*\right\|_\varphi = E_n(f)_\varphi$. Then we can write the following inequality :

$$\left\| f-V_n(f)\right\|_\varphi$$
$$\leq\left\| f-T_n^*\right\|_\varphi+\left\|T_n^*-V_n(f)\right\|_\varphi$$
$$\leq c_{47}E_n(f)_\varphi+\left\|V_n(T_n^*-f,\cdot)\right\|_\varphi$$
$$\leq c_{48}E_n(f)_\varphi. \qquad (3.8)$$

Comparing the estimates (3.6), (3.7) and (3.8) we find that

$$n^{-2l}\left\|V_n^{(2l)}(f)\right\|_\varphi+\left\| f-V_n(f)\right\|_\varphi$$
$$\leq c_{49}\left(\Omega_\varphi^l\left(\frac{1}{n+1},V_n(f)\right)+E_n(f)_\varphi\right)$$
$$\leq c_{50}\left(\Omega_\varphi^l(\frac{1}{n+1}, f)+\Omega_\varphi^l(\frac{1}{n+1}, f-V_n(f))+E_n(f)_\varphi\right)$$
$$\leq c_{51}\Omega_\varphi^l(\frac{1}{n+1}, f). \qquad (3.9)$$

Taking into account the relations (3.5) and (3.9) we obtain estimation (2.16) of Theorem 3.1.

According to [4] there exists a constant $c_{52}>0$ such that

$$\left\| f - S_n(f) \right\|_\varphi \le c_{52} E_n(f)_\varphi . \tag{3.10}$$

The proof of the estimation (2.17) is obtained in analogy to proof of the estimation (2.16) using the inequality (3.10).
Theorem 3.1 isproved.

References

1. Akgün R. Inequalities for one sided approximation in Orlicz spaces, Hacet. J. Math. Stat., V.40, (2011), pp.231-240.
2. Akgün A. Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth, Georgian Math. J., V.18, (2011), pp.203-235.
3. Akgün R., Kokilasdhvili V. Refined estimates of trigonometric approximation for functions with generalized derivatives in weighted variable exponent Lebesgue spaces, Georgian Math. J., V.19, (2012), pp.611-626.
4. Akgün R., Yildirir Y.E. Convolution and Jackson inequalities in Musielak – Orlicz spaces, Turkish Journal of Mathematics, V.42, (2018), pp.2166-2185.
5. Alaouia M.K, Nabilah T., Altanjia M. On some new non – linear diffusionmodels for the image filtering, Appl. Anal., V.93, (2014), pp.269-280.
6. Butzer P.L, Nessel R.J. Fourier Analysis and Approximation, V.1, Bessel, Switzerland Birkhauser Verlag, (1971).
7. Chen Y., Levine S., Rao M., Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math., V.66, (2006), pp.1383-1406.
8. Cohen F. On the degree of approximationof a function by the partial sums o its Fourier series, T Am. Math. Soc., V.235, (1978), pp.35-74.
9. Devore R.A. and Lorentz G.G. Constructive Approximation, Springer, Verlag, (1993).
10. Garidi W. On approximation by polynomials in Orlicz spaces, Approx. Thery Appl., V.7, (1991), pp.97-110.
11. Gavriljuk V.T. Linear summability methods for Fourier series and best approximation, Ukrain. Math. Zh., V.15, N.4, (1963), pp.412-418 (in Russian).

12. Giannetti F., Passarelli di Napoli A. Regularity results for a new class of functionals with nonstandard growth conditions, J. Differ. Equations, V.254, (2013), pp.1280-1305.
13. Gwiazda A.S. Nonlinier parabolie problems in Musielak – Orlicz spaces, Nonlinear Anal., V.98, (2014), pp.48-65.
14. Guven A., Israfilov D.M. Approximation by Means of Fourier trigonometric series in weighted Orlicz spaces, Adv. Stud. Contemp. Math. (Kyundshang), V.19, N.2, (2009), pp.283-295.
15. Guven A., Israfilov D.M. Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$, J. Math. Inequal., V.4, (2010), pp.285-299.
16. Harjulehto P., Hastö P., Latvala V., Toivanen O. Critical variable exponent functionals in image restoration, Appl. Math. Lett., V.26, (2013), pp.56-60.
17. Israfilov D.M, Koilashvili V., Samko S. Approximation in weighted Lebesgue and Smirnov spaces with variable exponents, Proc, A. Razmadze Math. Inst., V.143, (2007), pp.25-35.
18. Israfilov D.M., Guven A. Approximation by trigonometric polynomials in weighted Orlicz spaces, Studia Math., V.174, N.2, (2006), pp.147-168.
19. Israfilov D.M, Oktay B. and Akgün R. Approximation in Smirnov-Orlicz classes, Glas. Math. Ser. III, V.40, (2005), pp.87-102.
20. Israfilov D.M., Akgün R. Approximation in weighted Smirnov- Orlicz classes, J. Math. Kyoto Univ., V.46, N.4, (2006), pp.755-770.
21. Israfilov D.M, Testici A. Simultaneous approximation in Lebesgue spaces with variable exponent, Proceedings of the Institute of Mathematics, National Academy of Sciences of Azerbaijan, V.44, N.1, (2018), pp.3-18.
22. Israfilov D.M., Testici A. Some inverse and simultaneous approximation theorems in weighted variable exponent Lebesgue spaces, Analysis Mathematica, V.44, N.4, (2018), pp.475-492.
23. Jafarov S. Z. Approximation by rational functions in Smirnov- Orlicz classes, J. Math. Anal. Appl. V.379, (2011), pp.870-877.
24. Jafarov S.Z. Approximation of functions by De la Vallee- Poussin sums in weighted Orlicz spaces, Arab. J. Math., V.5, (2016), pp.25-137.
25. Jafarov S.Z. On approximation in weighted Smirnov- Orlicz classes, Complex Var. Elliptic Equ., V.57, N.5, (2012), pp.567-577.

26. Jafarov S.Z. The inverse theorem of approximation of the function in Smirnov- Orlicz classes, Math. Ineq. Appl., V.12, N.4, (2012), pp.835-844.
27. Jafarov S.Z. Approximation of conjugate functions by trigonometric polynomials in weighted Orlicz spaces, J. Math. Ineq., V.7, N.2, (2013), pp.271-281.
28. Jafarov S.Z. Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz spaces, Hacet. J. Math. Stat., V.2, N.3, (2013), pp.259-268.
29. Kokilashvili V.M, Samko S.G. Operators of harmonic analysis in weighted spaces with non- standart growth, J. Math. Anal. Appl., V.352, (2009), pp.15-34.
30. Kokilashvili V.M. and Samko S.G. A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces, Proc. A. Razmadze Math. Inst., V.151, (2009), pp.132-138.
31. Kokilashvili V.M. An inverse theorem of the constructive theory of functions in Orlicz spaces, Soob. Akad. Nauk Gruzin SSR, V.37, (1965), pp.263-270 (in Russian).
32. Kokilashvili V.M. On approximation of periodic functions in Orlicz spaces, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom Phys. (1966), pp.77-80.
33. Kaminska A.W. Existence result fort he motion of several rigid bodies in an incompressible non- Newtonian fluid with growht conditions in Orlicz spaces, Nonlinearity, V.27, (2014) , pp.685-716.
34. Musielak J. Orlicz Spaces and Modular Spaces, Lecture Notes in Mathematics, Berlin, Germany: Springer, (1983).
35. Musielak J. Approximation in modular function spaces, Dedicated to Roman Taberski on the occasion of his 70 th birthday, Funct. Approximateo Comment Mat., V.25, (1997), pp. 45-57.
36. Ok J. Gradient estimates for elliptic equations with $L^{p(\cdot)}\, LogL$ growth, Cale Var Partial Dif., V.55, (2016), pp.1-30.
37. Ovsi le.Yu, Serdyuk A.S. Approximation of continuous periodic functions by de la Vallée Poussin sums, Zb. Pr. Inst. Mat. NAN Ukr., V.8, N.1, (2011), pp.151-161.
38. Prestin J. On the approximation by de la Vallée Poussin sums and interpolatory polynomials in Lipschitz norms, Anal. Math., V.14, (1987), pp.251-259.
39. Ponomarenko V.G. Approximation of periodic functions in a Orlicz space, Sibirsk. Math. J., V.7, (1966), pp.1337-1346. (in Russian)

40. Ramazanov A.-R.K. On approximation by polynomials and rational functions in Orlicz spaces, Anal. Math., V.10, (1984), pp.117-132.
41. Stechkin S.B. The approximation of periodic functions by FejÈr sums, Trudy Math Inst. Steklov, V.62, (1961), pp.48-60. (in Russian).
42. Stechkin S.B. On the approximation of periodic functions by de la ValléePoussin sums, Anal. Math., V.4, (1978), pp.61-74.
43. Sharapudinov I.I. Some problems in approximation theory in the spaces $L^{p(x)}(E)$, Anal. Math., V.33, (2007), pp.135-153. (in Russian).
44. Sharapudinov I.I. Approximation of smooth functions in $L_{2\pi}^{p(x)}$ by Valleée-Poussin means, Izv. Saratov Univ. Mat. Mekh. Inform., V.13, (2013), pp.45-49.
45. Sharapudinov I.I. Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials, Izv. Ross. Akad. Nauk. Ser. Mat., V.77, (2013), pp.197-224. (in Russian)
46. Sharapudinov I.I. On direct and inverse theorems of approximation theory in variable Lebesgue space and Sobolev spaces, Azerbaijan Journal of Math., V.4, N.1, (2014), pp.55-72.
47. Serdyuk A.S, Ovsi le Yu, Musienko A.P. Approximation of classes of analytic functions by de la Vallée-Poussin sums in uniform metric, Rend. Mat. Appl., V.32, (2012), pp.1-15.
48. Simonov B.V. and Tikhonov S.Yu. On embeddings of function classes deÖned by constructive characteristics, Approximation and probability (Bedlewo, Poland 2004), Banach Center Publ., Polish Acad. Sci., Warsaw, V.72, (2006), pp.285-307.
49. Simonov B.V., Tikhonov S.Yu, Embedding theorems in the constructive theory of approximations, Mat. Sb. V.199, N.9, (2008), pp.107-148 (in Russian); translation in Sb. Math. V.199, N.(9-10) (2008), pp.1367-1407.
50. Timan M.F. Best approximation of a function and linear methods of summing Fourier series, Izv. Akad. Nauk SSSR Ser: Math., V.29, (1965), pp.587- 604 (in Russian).
51. Timan M.F. Some linear summation processes for Fourier series and best apptroximation, Dokl. Akad. Nauk SSSR, V.145, (1962), pp.741-743.
52. Tsyganok I.I., A generalization of a therem of Jackson, Mat. Sb., V.71, (1966), pp.257-260 . (in Russian).
53. Vallée Poussin Ch La, Sur la meilleure approximation des fonctions d'une variable reelle par des expressions d'ordre donné, Comptes dendus de l'Académie des Sciences, V.166, N.4, (1918), pp.799-802.

54. Yildirir Y.E., Israfilov D.M. Simultaneous and converse approximation theorems in weighted Lebesgue spaces, Math. Inequal. Appl., V.14, N.2, (2011), pp.359-371.
55. Yildirir Y.E. Approximation of periodic functions in weighted Orlicz spaces, Glas. Mat., V.47, (2012), pp.401-413.
56. Yildirir Y.E., Israfilov, D.M. The properties of convolution type transforms in weighted Orlicz spaces, Glas. Mat., V.45, (2010), pp.461-474.
57. Zakharov A.A. Bound on deviations of continuous periodic functions from their De La Vallée-Poussin sums, Matem. Zametki, V.3, N.1, (1968), pp.77-84. (in Russian).